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Riemannian Curvature and the Classification of the Riemann and Ricci Tensors in Space-Time

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Some theorems proved by Thorpe concerning the connection between the critical point structure of the Riemannian (sectional) curvature function and the Petrov classification are extended. A similar function is defined whose critical point structure is connected with the algebraic classification of the Ricci tensor.

1. INTRODUCTION

A geometrical interpretation of the Petrov classification of gravitational fields in General Relativity theory has been given by Thorpe (1969), who considered the critical points of a certain real-valued function. Let M be a space-time manifold and let $T_p(M)$ denote the tangent space to M at p. If G is the four-dimensional (Grassman) manifold of all two-dimensional subspaces (2-spaces) of $T_p(M)$, let $\bar{G}$ be the (four-dimensional) open submanifold of G consisting of all nondegenerate (non-null) 2-spaces of $T_p(M)$. One now introduces the Riemannian (sectional) curvature function as a differentiable map σ_p: $\bar{G} \to \mathbb{R}$ which associates with each 2-space F in $\bar{G}$ its Riemannian curvature

$$\sigma_p(F) = \frac{R_{abcd}F^{ab}F^{cd}}{2g_{a[c}g_{d]b}F^{ab}F^{cd}} \tag{1.1}$$

where R_{abcd}, g_{ab}, and F_{ab} are the components, in some chart about p, of the Riemann tensor, the metric tensor and any simple bivector representing the nondegenerate 2-space F. Thorpe demonstrated a connection between the *critical points* of the function σ and the Petrov classification of Einstein spaces[1] (space-times in which the Ricci tensor is proportional to the metric

[1] Details of the Petrov classification may be found in the articles of Bel (1962), Sachs (1961), and Ehlers and Kundt (1962). A discussion of critical points of functions on manifolds is contained in Milnor's book (1963).

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TABLE 1

Number n of spacelike critical points of σ_p	0	1	3	∞
Petrov type	III	II nondegenerate	I nondegenerate	II degenerate (N) I degenerate (D)

tensor). In fact, Thorpe showed that for Einstein spaces, the Petrov type of the Riemann tensor and the number n of spacelike critical points of σ_p are connected as shown in Table 1.[2] That only spacelike critical points need be considered is a consequence of the fact that in Einstein spaces, complementary 2-spaces represented by a certain simple bivector and its dual have the same Riemannian curvature. This follows from the relation

$$*R^*_{abcd} = -R_{abcd} \tag{1.2}$$

which is equivalent to the Einstein space condition. The symbol * in the appropriate place denotes the duality operator.

The Petrov type O case is characterized by the condition that R_{abcd} is proportional to $g_{a[c}g_{d]b}$ or, alternatively, by the condition that σ_p be a constant function on $\bar{G}$.

Thorpe also showed that in an Einstein space, the function σ_p could be continuously extended to all *null* 2-spaces at p if and only if σ_p was a constant function on $\bar{G}$.

In this paper it will firstly be shown that one may define a differentiable function $\phi_p: \bar{G} \to \mathbb{R}$ whose critical point structure is connected in a similar way to the algebraic classification of the Ricci tensor as the function σ_p is to the Petrov classification of the Riemann tensor in an Einstein space.

Secondly, Thorpe's results concerning the continuous extension of the function σ_p to null 2-spaces will be extended. It will be shown that if σ_p can be continuously extended to a *single* null 2-space then it is necessarily a constant function on $\bar{G}$. Further, this result will be proved without the use of the Einstein space condition so that it applies quite generally to the Riemann tensor at p. A similar result will be proved for the function ϕ_p where it will be found that if ϕ_p can be continuously extended to a single null 2-space then it is necessarily a constant function (in fact the zero function) on $\bar{G}$ and the Ricci tensor is proportional to the metric tensor at p.

Finally, some discussion of the classification theorems and some concluding remarks will be given in the final section of the paper.

[2] The statement of the connection given in Thorpe's paper is incomplete in that the Petrov type N case is omitted.

2. THE CLASSIFICATION

One first introduces a tensor E which is entirely equivalent to the trace-free Ricci tensor $\tilde{R}$ where, in coordinates, $\tilde{R}_{ab} = R_{ab} - \frac{1}{4}Rg_{ab}$ with R_{ab} and R the components of the Ricci tensor and its contraction, respectively. The equivalence of E and $\tilde{R}$ is contained in the component relation

$$E_{abcd} = \tfrac{1}{2}(\tilde{R}_{ac}g_{bd} - \tilde{R}_{bc}g_{ad} + \tilde{R}_{bd}g_{ac} - \tilde{R}_{ad}g_{bc}) \tag{2.1}$$

The tensor E has all the algebraic symmetries of the Riemann tensor and vanishes at a point p if and only if R_{ab} is proportional to g_{ab} at p. It satisfies the relations

$$-E_{ac}{}^{c}{}_{b} = \tilde{R}_{ab}, \qquad {}^*E_{abcd} = -E^*_{abcd}, \qquad {}^*E^*_{abcd} = E_{abcd} \tag{2.2}$$

The tensor E may be used as the basis of a classification of the Ricci tensor (Cormack and Hall, 1979) which, because of the algebraic similarities between the tensor E and the Riemann tensor, is similar to the Petrov classification and yields a classification which is equivalent to the usual one based on Segré types. A summary of the resulting canonical forms is given in the Appendix.

One can use the tensor E to construct the following real-valued differentiable function ϕ_p on $\bar{G}$ at p:

$$\phi_p(F) = \frac{E_{abcd}F^{ab}F^{cd}}{2g_{a[c}g_{d]b}F^{ab}F^{cd}} \tag{2.3}$$

The function ϕ_p is independent of the representative simple bivector F_{ab} chosen for F. Also, if F and F^* are complementary 2-spaces of $T_p(M)$ then it easily follows from (2.2) that $\phi_p(F) = -\phi_p(F^*)$. This means that in the classification, only spacelike critical points need be considered.

To examine the critical point structure of ϕ_p, let F be a spacelike member of $\bar{G}$. If e_1, e_2, e_3, and e_4 constitute a pseudo-orthonormal Lorentz basis of $T_p(M)$ with e_1, e_2, and e_3 spacelike and e_4 timelike and such that e_1 and e_2 span F (so that $e_1 \wedge e_2$ is a representative bivector for F), then it follows from the general theory of Grassman manifolds (Brickell and Clarke, 1970) that one can construct a chart about F in G which is contained in $\bar{G}$ and is such that a general member F' of this chart to which are attached the coordinates (x_1, x_2, x_3, x_4) is spanned by vectors u and v in $T_p(M)$, where

$$u = e_1 + x_1e_3 + x_2e_4, \qquad v = e_2 + x_3e_3 + x_4e_4 \tag{2.4}$$

One can now readily evaluate $\phi_p(F')$ from (2.3) using local coordinates about p in the manifold M. In fact, one finds for F' the representative bivector with components

$$F'_{ab} = \overset{12}{F}_{ab} + x_3\overset{13}{F}_{ab} + x_4\overset{14}{F}_{ab} + x_1\overset{32}{F}_{ab} + x_2\overset{42}{F}_{ab} + (x_1x_4 - x_2x_3)\overset{34}{F}_{ab} \tag{2.5}$$

where

$$\overset{ij}{F} = e_i \wedge e_j \qquad (1 \leqslant i, j \leqslant 4)$$

and so

$$F_{ab} = \overset{12}{F}_{ab}$$

and[3]

$$F^*_{ab} = \overset{34}{F}_{ab}$$

Then

$$\phi_p(F') = \frac{E_{abcd}F'^{ab}F'^{cd}}{f(x_1, x_2, x_3, x_4)} \tag{2.6}$$

where

$$f(x_1, x_2, x_3, x_4) = (1 + x_1{}^2 - x_2{}^2)(1 + x_3{}^2 - x_4{}^2) - (x_1x_3 - x_2x_4)^2 \tag{2.7}$$

The condition that F be a critical point of ϕ_p is now found by considering the equations (with the usual abuse of notation)

$$\left(\frac{\partial \phi_p}{\partial x_i}\right)_{F'=F} = 0 \qquad (i = 1, 2, 3, 4) \tag{2.8}$$

These equations are easily evaluated using (2.5), (2.6), and (2.7) and turn out to be equivalent to the conditions

$$E_{abcd}F^{cd} = \alpha F_{ab} + \beta F^*_{ab} \qquad (\alpha, \beta \in \mathbb{R}) \tag{2.9}$$

However, since F is represented by a simple non-null bivector, $F_{ab}F^{*ab} = 0$ and $F^*_{ab}F^{*ab} \neq 0$. It then follows from the second equation in (2.2), the symmetry property $E_{abcd} = E_{cdab}$ and the usual property of the duality operator, that $\beta = 0$ and that $\phi_p(F) = \frac{1}{2}\alpha$. The resulting equation (2.9), bearing in mind (2.1), (2.2), and the fact that F_{ab} is a simple bivector, is then seen to be the necessary and sufficient condition that the 2-space F is an invariant 2-space of the Ricci tensor (Cormack and Hall, 1979), that is if k^a are the components of any member of F then the vector with components $R^a{}_bk^b$ is also in F.

Now the Ricci tensor when considered in the usual way as a linear map $T_p(M) \rightarrow T_p(M)$ may take one of only four possible Segré types, {1, 1, 1, 1}, {2, 1, 1}, {3, 1}, and $\{z, \bar{z}, 1, 1\}$, together with their degeneracies (Plebanski, 1964; Hall, 1976). Also, a simple connection exists between the Segré type of the Ricci tensor and its invariant 2-space structure (Cormack and Hall, 1979) and a summary of this connection is given in the Appendix. From these results and those of the previous paragraph, one can construct a table (Table

[3] The orientation of the basis is chosen so that this dual condition holds.

TABLE 2

Number n of spacelike critical points of ϕ_p	0	1	3	∞
Segré type	{3, 1} {(3, 1)}	{2, 1, 1} $\{z, \bar{z}, 1, 1\}$ {2, (1, 1)} $\{z, \bar{z}, (1, 1)\}$	{1, 1, 1, 1}	all degeneracies of {1, 1, 1, 1} {(2, 1), 1} {(2, 1, 1)}

2) which gives the connection between the number n of spacelike critical points of ϕ_p and the Segré type of the Ricci tensor. The trivial case here is characterized either by the constancy of ϕ_p on $\bar{G}$, the condition that $E_{abcd} = 0$ at p, or the condition that R_{ab} is proportional to g_{ab} at p [see equations (2.1) and (2.2) at the end of Section 3].

Since F is a spacelike invariant 2-space of the Ricci tensor, it follows that F contains two orthogonal spacelike eigenvectors of the trace-free Ricci tensor (Hall, 1976[4]). The sum of the corresponding eigenvalues is equal to the real number α in (2.9).

3. CONTINUOUS EXTENSIONS OF THE RIEMANNIAN CURVATURE FUNCTION

In his paper, Thorpe showed that the function σ_p could be extended continuously to all null 2-spaces at p if and only if σ_p was a constant function on $\bar{G}$, or alternatively if and only if R_{abcd} was proportional to $g_{a[c}g_{d]b}$ at p. In fact a stronger result can be proved. The function σ_p need only be continuously extendible to a *single* null 2-space at p for it to be necessarily a constant function on $\bar{G}$ and the proof makes no use of the Einstein space condition at p and so holds for a general Riemann tensor. To see this, let σ_p be continuously extendible to the null 2-space F at p which is represented by the bivector $l \wedge x$, where l is a null vector and x a unit spacelike vector orthogonal to l. Then the condition that the real sequence $\{\sigma_p(F_n)\}$ be convergent, where F_n is, in turn, the sequence of non-null 2-spaces at p represented by the non-null simple bivectors $(l + t_n y) \wedge x$, $l \wedge (x + t_n m)$, and $l \wedge (x + t_n y + t_n m)$, with m a null vector and y a unit spacelike vector which together satisfy the relations $l\cdot m = 1$, $m\cdot x = y\cdot l = y\cdot x = y\cdot m = 0$ and $\{t_n\}$ a sequence of nonzero real numbers convergent to zero, yields the component relations at p

$$R_{abcd}F^{ab}F^{cd} = R_{abcd}F^{ab}W^{cd} = R_{abcd}F^{ab}W^{*cd} = R_{abcd}F^{ab}F^{*cd} = 0 \quad (3.1)$$

[4] In result (iii) on p. 542 of this reference, the word "distinct" should be replaced by the word "orthogonal."

where F_{ab} and W_{ab} are the components of the bivectors $2^{1/2}l \wedge x$ and $2l \wedge m$, respectively. Equation (3.1) implies that

$$R_{abcd}F^{cd} = \alpha F_{ab} + \beta F^{*}_{ab} \qquad (\alpha, \beta \in \mathbb{R}) \tag{3.2}$$

Further, a consideration of the limits of the above sequences $\{\sigma_p(F_n)\}$ reveals that

$$R_{abcd}W^{ab}W^{cd} + R_{abcd}W^{*ab}W^{*cd} = 0 \tag{3.3}$$

Next, one considers the sequence of non-null 2-spaces $\{F_n\}$ represented by the bivectors $(l + s_n m + t_n y) \wedge x$, where $\{s_n\}$ and $\{t_n\}$ are sequences of real numbers convergent to zero, arbitrary except for the condition that the resulting bivectors are non-null, which is $2s_n + t_n{}^2 \neq 0$ for each n. The convergence of $\{\sigma_p(F_n)\}$ in this case yields the component relations

$$R_{abcd}N^{ab}N^{cd} = R_{abcd}N^{ab}W^{*cd} = 2R_{abcd}N^{ab}F^{cd} - R_{abcd}W^{*ab}W^{*cd} = 0 \tag{3.4}$$

where N_{ab} are the components of the bivectors $2^{1/2}m \wedge x$. In a similar fashion, the sequences of non-null 2-spaces represented by the bivectors $(l + s_n m) \wedge (x + t_n m)$ and $(l + s_n m) \wedge (x + t_n y)$ may next be used. Here, $\{s_n\}$ and $\{t_n\}$ are real sequences convergent to zero restricted only by the conditions that for each n, $2s_n - t_n{}^2 \neq 0$ in the first case and $s_n \neq 0$ in the second. The resulting component relations are

$$R_{abcd}N^{ab}W^{cd} = R_{abcd}F^{*ab}F^{*cd} = 0 \tag{3.5}$$

Finally, consideration of more general sequences represented by bivectors of the form $(l + s_n y + t_n m) \wedge (x + p_n y + q_n m)$, where $\{s_n\}$, $\{t_n\}$, $\{p_n\}$, and $\{q_n\}$ are appropriately chosen real sequences convergent to zero, yields the final relations

$$2R_{abcd}F^{ab}N^{*cd} = 2R_{abcd}F^{*ab}N^{cd} = -R_{abcd}W^{ab}W^{*cd} \tag{3.6}$$

$$R_{abcd}N^{ab}F^{cd} + R_{abcd}N^{*ab}F^{*cd} = 0 \tag{3.7}$$

$$\begin{aligned} R_{abcd}N^{ab}N^{*cd} &= R_{abcd}W^{ab}N^{*cd} = R_{abcd}W^{*ab}N^{*cd} = R_{abcd}N^{*ab}N^{*cd} \\ &= R_{abcd}F^{*ab}W^{cd} = R_{abcd}F^{*ab}W^{*cd} = 0 \end{aligned} \tag{3.8}$$

Thinking of the Riemann tensor at p as having 21 components governed by the single linear relation $R_{a[bcd]} = 0$, one sees from the equations of this section that these 21 components are determined by two arbitrary real numbers and from the real and imaginary parts of the bivector completeness relation given in the Appendix, one can see that the Riemann tensor at p can be expressed as

$$R_{abcd} = Ag_{a[c}g_{d]b} + B(-g)^{1/2}\epsilon_{abcd} \tag{3.9}$$

where $g = \det g_{ab}$, ϵ_{abcd} is the alternating symbol and $A, B \in \mathbb{R}$. The condition $R_{a[bcd]} = 0$ then implies that $B = 0$ and so the Riemann tensor has the required form at p and σ_p is a constant function on $\bar{G}$.

If the Einstein space condition holds at p, then the above proof can be simplified by virtue of the condition (1.2) connecting the values of σ_p on dual 2-spaces.

A similar proof also shows that if the function ϕ_p can be extended continuously to a single null 2-space, then at p, E_{abcd} necessarily takes the same form as that given in (3.9) for R_{abcd}. Again the above proof may be simplified on account of the duality conditions in (2.2). Then the conditions $E_{a[bcd]} = 0$ and $E_{ab}{}^{ab}(= \tilde{R}_a{}^a) = 0$ give $A = B = 0$. Thus $E_{abcd} = 0$ and this is equivalent to the Ricci tensor being proportional to the metric tensor at p. Hence ϕ_p is a constant function (the zero function) on $\bar{G}$.

4. DISCUSSION AND CONCLUDING REMARKS

Suppose now that the Einstein space condition holds at p and that the Petrov type is nontrivial there. It is clear from Table I that the existence of infinitely many spacelike critical points of σ_p does not uniquely determine the Petrov type at p. Similar problems arise with the function ϕ_p in the determination (to within degeneracy) of the Segré types of the Ricci tensor in the cases $n = 1$ and $n = \infty$ of Table II. Considering the function σ_p first, the following result (Hall, 1978) shows how a simple condition on σ_p may be used to distinguish between the Petrov types for the case $n = \infty$. Let $S(l)$ denote the two-dimensional submanifold of $\bar{G}$ consisting of spacelike 2-spaces orthogonal to the null vector $l \in T_p(M)$. The members of $S(l)$ have the physical interpretation of being the family of wave surfaces of l—the totality of instantaneous wave surfaces to l of all observers with all possible velocities at p. The result (more precisely a special case of it) then states that if the Einstein space condition holds at p and if the Petrov type is nontrivial there, then σ_p is constant on $S(l)$ if and only if the Riemann tensor is algebraically special in the Petrov classification at p with l as a repeated principal null direction ($l^a l^c R_{abcd} \propto l_b l_d$). Since the two Petrov types N and D which occur in the case $n = \infty$ have exactly one and exactly two repeated principal null directions, respectively, and since σ_p determines principal null directions according to the above result, the cases are distinguished.

Similar results hold for the function ϕ_p. In fact, only slight modifications are required in the above result for it to say that ϕ_p is constant on $S(l)$ if and only if l is a Ricci eigenvector at p. To see this one first shows that the constancy of ϕ_p on $S(l)$ is equivalent to the condition $\overset{+}{E}_{abcd} V^{cd} \propto \bar{V}_{ab}$ at p, where $\overset{+}{E}_{abcd} = E_{abcd} + iE^*_{abcd}$ and $V_{ab} = F_{ab} + iF^*_{ab}$. This is then equivalent to l being a null Ricci eigenvector at p (Cormack and Hall, 1979). So ϕ_p can determine null Ricci eigenvectors and the discussion is completed by noting that in the cases $n = 1$ and $n = \infty$, Ricci tensors with Segré type $\{z, \bar{z}, 1, 1\}$ or its degeneracy have no null eigendirections, those with Segré type $\{2, 1, 1\}$ or its degeneracies have a unique null eigendirection, and those with Segré type $\{1, 1, 1, 1\}$ or its degeneracies have either no or at least two null eigendirections (see the Appendix).

One final point can now be discussed. A function which is reminiscent of the functions σ_p and ϕ_p above was considered many years ago by Eisenhart and was based on the work of Ricci (see, for example, Eisenhart, 1966[5]). This function, which shall be denoted by θ_p, is a real-valued differentiable function on the one-dimensional subspaces of $T_p(M)$ where now M is an n-dimensional *positive definite* Riemannian manifold. It is defined in components at p by

$$\theta_p(K) = \frac{R_{ab}k^a k^b}{g_{ab}k^a k^b} \tag{4.1}$$

where k is any nonzero member of the one-dimensional subspace K and where g_{ab} are the components of the positive definite metric at p. That θ_p is independent of the choice of nonzero member of K is clear. So θ_p is a differentiable map: $\mathbb{P}(n-1, \mathbb{R}) \to \mathbb{R}$ and since the $(n-1)$-dimensional real projective space $\mathbb{P}(n-1, \mathbb{R})$ is a manifold diffeomorphic to the Grassman manifold $G(1, n)$, one may choose the usual Grassman coordinates about k. An orthonormal set of vectors at p, $e_1, e_2, \ldots, e_n$ is selected with $k = e_1$ such that the members of a chart about k are coordinatized by the $(n-1)$-tuple $(x_1, x_2, \ldots, x_{n-1})$ if they are spanned by the vector $e_1 + x_1 e_2 + \cdots + x_{n-1}e_n$. The critical points of θ_p may now be examined by the same method as that used previously in this paper and one readily achieves Eisenhart's result that the critical points and associated critical values of θ_p are precisely the eigendirections and associated eigenvalues of the Ricci tensor at p.

If M is a space-time, one might consider the function in (4.1) defined for non-null directions only and ask if it can be continuously extended to null directions and if Eisenhart's theorem can be generalized. It turns out that if θ_p can be continuously extended to a single null direction represented, say, by the null vector $l \in T_p(M)$, then necessarily the Ricci tensor is proportional to the metric tensor at p and so θ_p is a constant function. To see this, one considers sequences of non-null directions spanned by vectors of the form $l + t_n\xi$, where ξ is, in turn, the vector x, y, $\frac{1}{2}(x + y)$, and m, where $\{t_n\}$ is a

[5] This result disagrees with a theorem in Section 33 of the book by Eisenhart (1966) in much the same way as Thorpe (1969) disagrees with a similar theorem for Riemann tensors in the book of Petrov (1969). Also, the classification scheme for symmetric tensors in the same section of Eisenhart's book is incomplete in that the case of complex eigenvalues is not considered.

sequence of nonzero real numbers convergent to zero, and where the general notation of Section 3 is used. This gives the following relations at p:

$$\begin{gathered} R_{ab}l^a x^b = R_{ab}l^a y^b = R_{ab}l^a l^b = R_{ab}x^a y^b = 0 \\ R_{ab}x^a x^b = R_{ab}y^a y^b = R_{ab}l^a m^b \end{gathered} \tag{4.2}$$

Finally, consideration of a sequence of non-null directions spanned by vectors of the form $l + s_n m + t_n \xi$ where ξ is, in turn, the vector x and y and where $\{s_n\}$ and $\{t_n\}$ are sequences of real numbers convergent to zero, arbitrary except for the restriction $t_n^2 + 2s_n \neq 0$ for each n, yields at p

$$R_{ab}m^a m^b = R_{ab}m^a x^b = R_{ab}m^a y^b = 0 \tag{4.3}$$

Equations (4.2) and (4.3) imply that

$$R_{ab} = \tfrac{1}{4}R(2l_{(a}m_{b)} + x_a x_b + y_a y_b) = \tfrac{1}{4}Rg_{ab} \tag{4.4}$$

where the completeness relation at p for the null tetrad has been used. This is the required result.

For a space-time then either θ_p is a constant function, or the critical points of θ_p are precisely the spacelike and timelike eigendirections of the Ricci tensor.

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APPENDIX

Here, a brief summary of the use of the E tensor in the classification of the Ricci tensor will be given. Further details may be found in Cormack and Hall (1979). Let $(l, m, t, \bar{t})$ be a complex null tetrad at p with $l \cdot m = t \cdot \bar{t} = 1$ the only nonvanishing inner products between the tetrad members and let the associated complex bivectors be given in components by $V_{ab} = 2l_{[a}\bar{t}_{b]}$, $U_{ab} = 2m_{[a}t_{b]}$, and $M_{ab} = 2l_{[a}m_{b]} + 2\bar{t}_{[a}t_{b]}$. The tetrad is assumed oriented so that these complex bivectors and the real bivectors used in this paper are connected by the relations $V_{ab} = F_{ab} + iF^*_{ab}$, $U_{ab} = N_{ab} + iN^*_{ab}$, and $M_{ab} = W_{ab} + iW^*_{ab}$. Also, one has the completeness relation

$$g_{a[c}g_{d]b} + \tfrac{1}{2}i(-g)^{1/2}\epsilon_{abcd} = V_{ab}U_{cd} + U_{ab}V_{cd} - \tfrac{1}{2}M_{ab}M_{cd}$$

The four algebraic (Segré) types for the Ricci tensor may now be described.

Type $\{z, \bar{z}, 1, 1\}$ and its Degeneracy. In this case, the null tetrad at p can be chosen so that

$$E_{abcd} = \mathrm{Re}\,\{A_1(\overline{U}_{ab}U_{cd} - \overline{V}_{ab}V_{cd}) + A_2\overline{M}_{ab}M_{cd} + A_3(\overline{U}_{ab}V_{cd} + \overline{V}_{ab}U_{cd})\} \qquad (A_1 \neq 0)$$

$$R_{ab} = \alpha_1(l_al_b - m_am_b) + 2\alpha_2 l_{(a}m_{b)} + \alpha_3 x_a x_b + \alpha_4 y_a y_b \qquad (\alpha_1 \neq 0)$$

where here and throughout the Appendix, capital Latin letters and Greek letters denote real numbers and $2^{1/2}t_a = x_a + iy_a$.

In this type, a unique spacelike invariant 2-space is admitted. The Ricci tensor admits no null eigenvectors.

Type $\{1, 1, 1, 1\}$ and its Degeneracies. In this case, the null tetrad at p can be chosen so that

$$E_{abcd} = \mathrm{Re}\,\{B_1(\overline{U}_{ab}U_{cd} + \overline{V}_{ab}V_{cd}) + B_2\overline{M}_{ab}M_{cd} + B_3(\overline{U}_{ab}V_{cd} + \overline{V}_{ab}U_{cd})\}$$

$$R_{ab} = \beta_1(l_al_b + m_am_b) + 2\beta_2 l_{(a}m_{b)} + \beta_3 x_a x_b + \beta_4 y_a y_b$$

Here, the Ricci tensor is diagonalizable and if the four eigenvalues $(\beta_2 \pm \beta_1)$, β_3, and β_4 are all distinct, exactly three spacelike invariant 2-spaces are admitted. Otherwise there are infinitely many spacelike invariant 2-spaces. Such Ricci tensors admit either no or at least two null eigendirections.

Type $\{2, 1, 1\}$ and its Degeneracies. In this case, the null tetrad may be chosen so that

$$E_{abcd} = \mathrm{Re}\,\{C_1\overline{V}_{ab}V_{cd} + C_2\overline{M}_{ab}M_{cd} + C_3(\overline{U}_{ab}V_{cd} + \overline{V}_{ab}U_{cd})\} \qquad (C_1 \neq 0)$$

$$R_{ab} = 2\gamma_1 l_{(a}m_{b)} + \gamma l_a l_b + \gamma_2 x_a x_b + \gamma_3 y_a y_b \qquad (\gamma \neq 0)$$

The cases $\{2, 1, 1\}$ ($\gamma_1, \gamma_2, \gamma_3$ all distinct) and $\{2,(1, 1)\}$ ($\gamma_2 = \gamma_3 \neq \gamma_1$) give a unique spacelike invariant 2-space. The other degeneracies give infinitely many spacelike invariant 2-spaces. The Ricci tensor admits the unique null eigendirection l^a.

Type $\{3, 1\}$ and its Degeneracies. In this final case, the null tetrad may be chosen so that

$$E_{abcd} = \mathrm{Re}\,\{D_1(\overline{M}_{ab}M_{cd} - 2\overline{U}_{ab}V_{cd} - 2\overline{V}_{ab}U_{cd}) + D_2(\overline{V}_{ab}M_{cd} + \overline{M}_{ab}V_{cd})\} \qquad (D_2 \neq 0)$$

$$R_{ab} = 2\delta_1 l_{(a}m_{b)} + 2\delta l_{(a}x_{b)} + \delta_1 x_a x_b + \delta_2 y_a y_b \qquad (\delta \neq 0)$$

In this case, no spacelike invariant 2-spaces are admitted. The Ricci tensor admits the unique null eigendirection l^a.

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